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Plane contact problem for a half-space with boundary imperfections

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Abstract

In this paper the new micrmodelling approach to the contact problem for a half-space with boundary imperfections is proposed. The approach is based on a periodic distribution of micro-undulations along the space boundary and leads to the 2-D mathematical macro-model of the contact problem. The general idea of the modelling takes into account certain concepts used in the investigation of periodic composite materials (see e.g. Woźniak, 1993). The resulting model constitutes a generalization of the known Winkler-type model (see e.g. Shtayerman, 1949). The numerical solution to the special problem shows the boundary imperfections effect on the contact of bodies.

1. Introduction

The known contact problems are usually formulated under the assumption of the perfect contact interface. However, the experimental investigations carried out on the real body surfaces display the existence of imperfections which compose the boundary microgeometry. The influence of the boundary imperfections on the contact problem solution was studied firstly by Shtayerman (1949). It was supposed phenomenologically that the deformation of a surface micro-undulation is described by the Winkler model where the microgeometrical parameter has to be obtained experimentally. In this paper a new two-dimensional model for the deformation of a surface micro-imperfection is obtained in the framework of the analytical micromodelling procedure. The proposed model permits us to investigate analytically the microgeometry of the surface by means of a certain averaging procedure based on that applied in Woźniak (1993).

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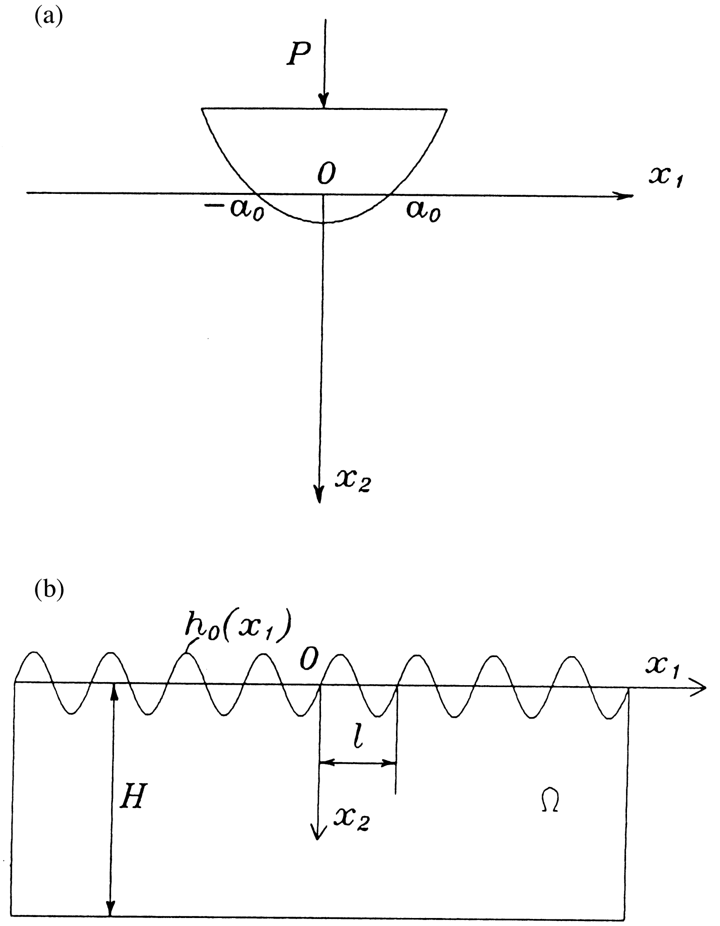

Fig. 1. (a) Geometry of the contact. (b) Model of the micro-folded subboundary layer.

2. Problem formulation

The geometry of the contact on a macro-level is shown in Fig. 1a. The rigid cylindrical punch of the radius R is pressed by a load P to the elastic half-space with boundary imperfections. The deformation in the half-space caused by the contact pressure $p(x_1)$ is assumed to be plane and hence the problem is considered in the Cartesian coordinates x_1x_2 where $x_2 > 0$ will be referred to as an elastic half-plane. The friction forces are neglected in the contact area $(-a_0, a_0)$ and the boundary of the half-plane for $|x_1| > a_0$ is free of the external loading.

Following Shtayerman (1949), we assume that the vertical displacement of the imperfected half-plane boundary may be presented in the form

$$u_2(x_1) = u_2^e(x_1) + u_2^a(x_1), \quad |x_1| < a_0 \quad x_2 = 0 \tag{1}$$

where $u_2^e(x_1)$ are elastic displacements produced by the contact pressure $p(x_1)$ and $u_2^a(x_1)$ are displacements due to boundary imperfections in the contact area.

The elastic displacements are obtained from the solution of the known stationary problem of the elasticity theory and have the following form (see e.g. Shtayerman, 1949)

$$u_2^e(x_1) = \frac{1-v}{\pi\mu}\int_{-a_0}^{a_0} p(\xi)[-\ln|x_1-\xi|+C]\,\mathrm{d}\xi, \quad |x_1| < a_0 \tag{2}$$

where v, μ are Poisson's ratio and the shear modulus, respectively, and C is an arbitrary constant.

The analytical form of the displacements $u_2^a(x_1)$ will be obtained in Section 3 by the proposed micromodelling approach to the problem of boundary imperfections in the contact area.

3. Micromodelling of the boundary imperfections

The aim of this Section is to obtain an averaged model of the deformation of the subsurface layer which takes into account the boundary microgeometry. To this end let us introduce a subboundary layer of the undulated half-plane bounded by a micro-periodically folded smooth boundary and the line $x_2 = H$, cf Fig. 1b. Hence this layer occupies the region

$$\Omega = \{(x_1, x_2) \colon |x_1| < \infty, \quad h_0(x_1) < x_2 < H\}$$

where $h_0(x_1)$ is the l-periodic function where l is sufficiently small compared with the half-width of the contact region a_0: $l \ll a_0$ as well as with the length dimension H. In the sequel the layer Ω with the smooth micro-periodically folded boundary (micro-undulated surface) will be treated as a micro-model of the imperfected boundary of the half-plane introduced in Section 2.

Let u_i, ε_{ij}, σ_{ij} be displacements, strains and stresses, respectively, and p_i be external forces applied to the part Π of the aforementioned microfolded boundary; here and in the sequel subscripts i, j, $k, \ldots$ run over 1, 2 and the summation convection holds. In the framework of the linear elasticity theory the stationary behaviour of the layer Ω is described by:

the strain–displacement relations:

$$\varepsilon_{ij} = u_{(i,j)} \tag{3}$$

the stress–strain relations:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{4}$$

where C_{ijkl} are components of the elastic modulate tensor, and the equilibrium equation assumed here in the form of condition

$$\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \int_\Pi p_i\delta u_i\,\mathrm{d}a \tag{5}$$

which holds for every specified virtual displacement field $\delta u_i = \delta u_i(x_1, x_2)$, such that $\delta u_i = 0$, for $x_2 = H$.

In order to perform the modelling procedure we introduce the concept of the l-macrofunction and that of the micro-shape function (see e.g. Woźniak, 1993). Let $F(\cdot)$ be a real valued function defined in Ω and ε_F stands for a certain small accuracy parameter related to the computation of the values of F. Define by $\|x'-x''\|$ the distance between points x', x'' in Ω. If for every two points $x', x'' \in \Omega$ such that $\|x'-x''\| < l$ function $F(\cdot)$ satisfies condition.

$$|F(x')-F(x'')| < \varepsilon_F$$

then it will be called the l-macro function. If $F(\cdot)$ satisfies the above condition together with all its derivatives then it will be called the regular l-macrofunction.

Let $h(\cdot)$ be l-periodic continuous function defined on $\mathbf{R}$ which for every $x_1 \in \mathbf{R}$ satisfies conditions

(i) $h(x_1) \in O(l)$,
(ii) $lh_{,1}(x_1) \in O(l)$,
(iii) $\langle h \rangle = 0$,

where

$$\langle f \rangle \equiv \frac{1}{l}\int_0^l f(x_1)\,\mathrm{d}x_1$$

is an averaged value of f over $(0, l)$. Under aforementioned condition function $h(\cdot)$ will be called the micro-shape function.

The proposed micromodelling procedure of the boundary imperfections will be based on the three following assumptions:

(1) Macro Kinematic Hypothesis states that the displacements in the region Ω can be assumed in the form

$$u_i(x_1, x_2) = U_i(x_1, x_2) + h(x_1)Q_i(x_1, x_2), \quad (x_1, x_2) \in \Omega \tag{6}$$

where U_i, Q_i are unknown regular l-macrofunctions, called macro-displacements and correctors, respectively, and $h(x_1)$ is the continuous l-periodic micro-shape function. This function have to be defined a priori in every problem under consideration.

(2) 2-D Modelling Hypothesis postulates the macrofunctions U_i, Q_i in the form

$$\begin{aligned} U_i(x_1, x_2) &= W_i(x_1)d(x_2) \\ Q_i(x_1, x_2) &= V_i(x_1)d(x_2) \end{aligned} \tag{7}$$

where W_i, V_i are arbitrary regular macrofunctions independent of x_2-coordinate and $d(x_2)$ is postulated a priori decay function defined on $(0, H)$ (see e.g. Vlasov and Leontiev, 1960). The function $d(x_2)$ is sufficiently regular and satisfies and conditions $d(0) = 1$, $d(H) = 0$, $d_{,2}(x_2) < 0$ for every $x_2 \in (0, H)$. Thus, the functions W_i, V_i are, respectively, macrodisplacements and correctors on the upper boundary of the layer.

(3) Macro Modelling Approximation permits in calculations of averages $\langle \cdot \rangle$ to neglect terms of an order ε_F as compared to values of an arbitrary l-macro function F. Under this assumption eqns (3) together with (6), (7) can be transformed to the following form

$$\varepsilon_{11} = (W_{1,1} + h_{,1}V_1)d + O(\varepsilon_v)$$
$$\varepsilon_{12} = \tfrac{1}{2}[(W_1 + hV_1)d_{,2} + (W_{2,1} + h_{,1}V_2)d] + O(\varepsilon_v)$$

$$\varepsilon_{22} = (W_2 + hV_2)d_{,2} + O(\varepsilon_v) \tag{8}$$

The macro modelling approximation makes it possible to neglect term $O(\varepsilon_F)$ in the formula

$$\int_\Omega fF\,\mathrm{d}\Omega = \langle f\rangle \int_\Omega F\,\mathrm{d}\Omega + O(\varepsilon_F) \tag{9}$$

which holds for any integrable l-periodic function f and continuous l-macrofunction F.

Combining formulae (4), (8) with the principle of virtual work (5) and applying the macro modelling approximation we obtain the governing equations of the 2-D model of the subsurface layer in the form

$$\begin{aligned} & R_i - S_{i,1} = \tilde{p}_i \\ & H_i = 0 \end{aligned} \tag{10}$$

where $\tilde{p}_i$ are averaged external loads (per length unit of $x_2 = 0$) and

$$\begin{bmatrix} R_i \\ S_i \\ H_i \end{bmatrix} = \begin{bmatrix} \langle C_{i2k2}a_{33}\rangle & \langle C_{i2k1}a_3\rangle & \begin{array}{c}\langle C_{i2k1}h_{,1}a_3\rangle + \\ + \langle C_{i2k2}ha_{33}\rangle\end{array} \\ \langle C_{i1k2}a_3\rangle & \langle C_{i1k1}a\rangle & \begin{array}{c}\langle C_{i1k1}h_{,1}a\rangle + \\ + \langle C_{i1k2}ha_3\rangle\end{array} \\ \begin{array}{c}\langle C_{i1k2}h_{,1}a_3\rangle + \\ + \langle C_{i2k2}ha_{33}\rangle\end{array} & \begin{array}{c}\langle C_{i1k1}h_{,1}a\rangle + \\ + \langle C_{i2k1}ha_3\rangle\end{array} & \begin{array}{c}\langle C_{i1k1}h_{,1}h_{,1}a\rangle + \\ + \langle C_{i1k2}hh_{,1}a_3\rangle + \\ + \langle C_{i2k1}hh_{,1}a_3\rangle + \\ + \langle C_{i2k2}hha_{33}\rangle\end{array} \end{bmatrix} \begin{bmatrix} W_k \\ W_{k,1} \\ V_k \end{bmatrix} \tag{11}$$

where we have donated

$$a(x_1) = \int_{h_0(x_1)}^{H} d^2(x_2)\,\mathrm{d}x_2,$$

$$a_3(x_1) = \int_{h_0(x_1)}^{H} d(x_2)d_{,2}(x_2)\,\mathrm{d}x_2,$$

$$a_{33}(x_1) = \int_{h_0(x_1)}^{H} d_{,2}(x_2)d_{,2}(x_2)\,\mathrm{d}x_2$$

Substituting the right-hand sides of (11) into formulae (10) we arrive at the system of three differential equations for the macrofunctions W_i and three algebraic equations for V_i. Since $l \ll H$ then in calculations of averages in eqns (11) we can neglect terms involving the length parameter l as small compared to terms independent of l. Thus, the final form of the model equations will be given by

$$\tilde{R}_i - \tilde{S}_{i,1} = \tilde{p}_i$$

$$\tilde{H}_i = 0 \tag{12}$$

where

$$\begin{bmatrix} \tilde{R}_i \\ \tilde{S}_i \\ \tilde{H}_i \end{bmatrix} = \begin{bmatrix} \langle C_{i2k2}a_{33}\rangle & \langle C_{i2k1}a_3\rangle & \langle C_{i2k1}a_3h_{,1}\rangle \\ \langle C_{i1k2}a_3\rangle & \langle C_{i1k1}a\rangle & \langle C_{i1k1}ah_{,1}\rangle \\ \langle C_{i1k2}a_3h_{,1}\rangle & \langle C_{i1k1}ah_{,1}\rangle & \langle C_{i1k1}ah_{,1}h_{,1}\rangle \end{bmatrix} \begin{bmatrix} W_k \\ W_{k,1} \\ V_k \end{bmatrix} \tag{13}$$

Let us observe the boundary surface with micro-undulations, shown in Fig. 1b, is smooth; this way the friction forces in the contact area $(-a_0, a_0)$ will be neglected and the problem becomes similar to that investigated by Shtayerman (1949). Moreover, following the approach used in Vlasov and Leontiev (1960) we shall assume that under the normal pressure the horizontal component of displacements can be neglected as small compared to the normal component. It means that in the subsequent analysis we shall consider the simplified model in which $W_1 = V_1 = 0$, but $W_2 \neq 0$, $V_2 \neq 0$. Simplified models of this kind have found many applications mainly in structure–subsoil interaction problems (see e.g. Vlasov and Leontiev, 1960). We also assume that the layer is homogeneous, i.e.

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}) + \lambda\delta_{ij}\delta_{kl}$$

where δ_{ij} is Kroneker symbol, λ and μ are Lamé constants.

In this case after modifying assumptions (7) by means $W_1 = V_1 = 0$, and applying the micro-modelling procedure similar to that leading to (12), (13), we obtain the following differential equation for the macrofunction W_2

$$2tW_{2,11} - kW_2 + p_2 = 0 \tag{14}$$

where

$$t = \frac{\mu}{2}\left[\langle a\rangle - \frac{\langle ah_{,1}\rangle^2}{\langle a(h_{,1})^2\rangle}\right], \quad k = (\lambda + 2\mu)\langle a_{33}\rangle \tag{15}$$

It can be seen that equation (14) from a formal point of view is similar to the equation of the Vlasov model of an elastic subsoil layer (see e.g. Vlasov and Leontiev, 1960, p. 30). The above equation together with definitions (15) describes on the macro-level the effect of boundary micro-undulations on contact pressure distribution. This effect depends on the shape of micro-undulations because coefficient t depends on the derivatives of function $h(\cdot)$. On the other hand from the definition of t and assumption $lh_{,1}(x_1) \in O(l)$ it follows that the value of this coefficient is independent of l. We deal here with the method of micromodelling similar to that used in the homogenization of periodic composite materials, where the resulting macro-model are also independent of the cell size (see e.g. Jikov et al., 1994).

The solution of eqn (14) can be assumed in the form

$$W_2(x_1) = \frac{1}{4\alpha t}\int_{-\infty}^{\infty} p_2(\xi)K(\xi - x_1)\,\mathrm{d}\xi \tag{16}$$

where

$$\alpha = \sqrt{\frac{k}{2t}}$$

$$K(z) = \begin{cases} e^{-\alpha z}, & z \geqslant 0 \\ e^{\alpha z}, & z < 0 \end{cases}$$

If

$$p_2(x_1) = \begin{cases} p(x_1), & |x_1| \leqslant \alpha_0 \\ 0, & |x_1| > \alpha_0 \end{cases}$$

then

$$u_2^a(x_1) = W_2(x_1) = \frac{1}{4\alpha t}\int_{-a_0}^{a_0} p(\xi)K(\xi - x_1)\,\mathrm{d}\xi \tag{17}$$

The obtained model is characterized by two coefficients k and t which will be called the microgeometry parameters. The phenomenological model of the boundary imperfections which was presented in Shtayerman (1949) can be formally derived from results obtained above by setting $t = 0$. In this case

$$u_2^a(x_1) = \frac{1}{k}p(x_1), \quad |x_1| \leqslant a_0 \tag{18}$$

where k has the form (15). It has to be emphasized that the microgeometry parameters k, t are given here by the explicit analytical formulas (15) where t describes a new effect (postponed in Shtayerman, 1959) of the deformed boundary curvature on the contact force.

4. Integral equation of the contact problem

By satisfying the boundary condition

$$u_{2,1}(x_1) = -\frac{x_1}{R}, \quad |x_1| < a_0 \tag{19}$$

by the formulae (1), (2) and (17) we obtain the following singular integral equation of the problem

$$\frac{1}{4t}\int_{-a_0}^{a_0} p(\xi)K_1(x_1 - \xi)\,\mathrm{d}\xi + \frac{1-\nu}{\pi\mu}\int_{-a_0}^{a_0} \frac{p(\xi)}{x_1 - \xi}\,\mathrm{d}\xi = -\frac{x_1}{R}, \quad |x_1| < a_0 \tag{20}$$

where the kernel $K_1(z)$ have the form

$$K_1(z) = \begin{cases} e^{\alpha z}, & z < 0 \\ -e^{-\alpha z}, & z \geqslant 0 \end{cases}$$

The integral equation (2) should be solved together with the equilibrium condition

$$\int_{-a_0}^{a_0} p(\xi)\,\mathrm{d}\xi = P \tag{21}$$

and the physical condition for the determination of the contact area

$$p(\pm a_0) = 0 \tag{22}$$

Introducing dimensionless variables and functions

$$s = x_1/a_0, \quad \eta = \xi/a_0, \quad p^*(s) = \frac{a_0 p(x_1/a_0)}{P}$$

the integral eqn (20) and conditions (21), (22) can be rewritten to the new form

$$\beta_0 \int_{-1}^{1} p^*(\eta) K_1^*(s-\eta)\,\mathrm{d}\eta + \frac{1}{\pi}\int_{-1}^{1} \frac{p^*(\eta)}{s-\eta}\,\mathrm{d}\eta = -\frac{2}{\pi}\frac{a_0^2}{a_H^2}\frac{P_H}{P}s, \quad |s| < 1 \tag{23}$$

$$\int_{-1}^{1} p^*(\eta)\,\mathrm{d}\eta = 1 \tag{24}$$

$$p^*(\pm 1) = 0 \tag{25}$$

where

$$K_1^*(z) = \begin{cases} \mathrm{e}^{z/\alpha_0}, & z < 0 \\ -\mathrm{e}^{-z/\alpha_0}, & z \geqslant 0 \end{cases}, \quad \alpha_0 = \frac{1}{\alpha_0}\sqrt{\frac{2t}{k}}, \quad \beta_0 = \frac{a_0}{4t}\frac{\mu}{1-\nu}$$

By means of (15) the shape of micro-undulations is given by parameters α_0, β_0 which describe in the averaged form (on a macro-level) the effect of boundary imperfections on contact pressure distribution. These parameters can be calculated for every material after specifying function $h(\cdot)$.

The half-width of the contact area a_H and the load P_H in the corresponding problem for the half-space are connected by the relation (see e.g. Timoshenko and Goodier, 1934)

$$a_H^2 = \frac{2P_H R}{\pi}\frac{1-\nu}{\mu}$$

The value of a_0 in eqn (23) is unknown and may be obtained by the iteration procedure from the extra conditions (25). We can also apply simplified approach in which the width of the contact area is assumed to be equal to that of the Hertz problem, i.e. $a_0/a_H = 1$, but the load P related to this contact zone is unknown. In this case the integral equation (23) and the condition (24) are sufficient to determine the contact pressure $p^*(s)$ and the ratio P_H/P.

5. Numerical solution of the integral equation

Equation (23) is a Cauchy-type singular integral equation for the contact pressure. The function $p^*(s)$ may be taken in the form

$$p^*(s) = \sqrt{1-s^2}\varphi(s), \quad |s| \leqslant 1 \tag{26}$$

where $\varphi(s)$ is a regular unknown function.

For the numerical solution of the system of the singular integral equations (23), (24) we use the Gauss–Chebyshev quadrature method (see e.g. Belotserkovskij and Lifanov, 1985). The discretized forms of these equations are

$$\gamma_{0n} + \frac{1}{\pi}\sum_{k=1}^{n}\frac{\varphi(\eta_k)w_k}{s_m-\eta_k} + \beta_0\sum_{k=1}^{n}\varphi(\eta_k)w_kK_1(s_m-\eta_k) = -\frac{2}{\pi}\frac{P_H}{P}s_m, \quad m = 1,\ldots,n+1 \tag{27}$$

$$\sum_{k=1}^{n}\varphi(\eta_k)w_k = 1 \tag{28}$$

where

$$\eta_k = \cos\left[\frac{k\pi}{n+1}\right], \quad w_k = \frac{\pi}{n+1}\sin^2\left[\frac{k\pi}{n+1}\right], \quad k = 1,\ldots,n$$

$$s_m = \cos\left[\frac{2m-1}{2n+2}\right], \quad m = 1,\ldots,n+1$$

The regularized parameter γ_{0n} is introduced in the system of linear algebraic eqns (27), (28). It is known (see e.g. Belotserkovskij and Lifanov, 1985) that the condition

$$\lim_{n\to\infty}\gamma_{0n} = 0 \tag{29}$$

provides a unique solution. In the numerical procedure the condition (29) serves for the determination of the number n.

The system of $n+2$ linear algebraic eqns (27), (28) is sufficient to find the $n+2$ unknowns: $\varphi(\eta_k)$, $k = 1,\ldots,n$; γ_{0n}, P_H/P.

6. Results

The aim of the numerical analysis is to display the roughness effect on the solution of the contact problem. The dimensionless coefficients α_0, β_0 are independent parameters of the problem; we have explained in Section 3 that they depend on the shape of boundary micro-undulations but are independent of their size.

Figure 2 shows the dependence of the ratio P/P_H with the change of the microgeometry parameter α_0 for some values of the second coefficient β_0. In the case of the plane boundary the ratio P/P_H is equal to one. This result is obtained at $\alpha_0 \to 0$ for all values of the parameter β_0. It was shown that the ratio P/P_H decreases with the parameter α_0. This effect is greater for the increasing of the parameter β_0. Thus, the force P which is needed to obtain the contact region $a = a_H$ in the case of micro-undulated boundary is smaller than that in the case of plane boundary. It is clear that the contact area in the considered problem is greater than Hertz contact zone.

The distribution of the dimensionless contact pressure $p^*(s)$ in the contact region $(-1, 1)$ is

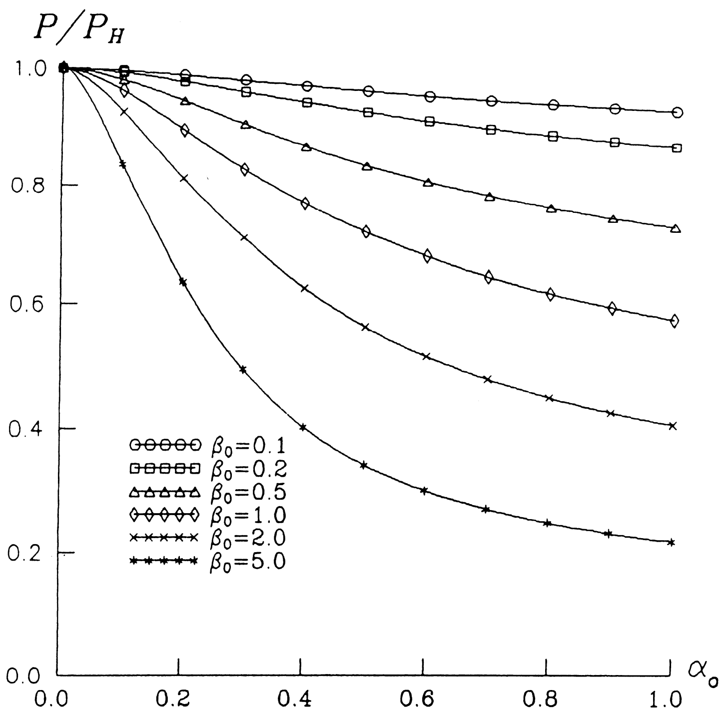

Fig. 2. The effect of parameter α_0 on ratio P/P_H for different values of coefficient β_0.

presented in Fig. 3 for $\beta_0 = 2.0$ and for four values of the parameter α_0. The line $\alpha_0 = 0$ corresponds to the Hertz distribution. The growth of the roughness parameter α_0 leads to the decreasing of the maximum value of the contact pressure in the center of the contact area.

7. Conclusions

Two main features of the proposed micro-undulated contact zone model can be mentioned. First, it is based on the micromodelling procedure and hence all coefficients can be obtained analytically by the averaging of the microgeometry of the contact area. Second, it was shown that the problem on the macro-level is affected also by the curvature of the deformed contact surface since the governing eqn (14) we deal with the term $2tw_{2,11}$. Hence the main conclusion of the paper is that the proposed two-parameter model (14) constitutes a certain generation of the known Winkler-type model of the contact problem introduced in Shtayerman (1949). Moreover, the obtained solution to the contact problem shows that the model proposed can be applied to the investigation of special engineering problems.

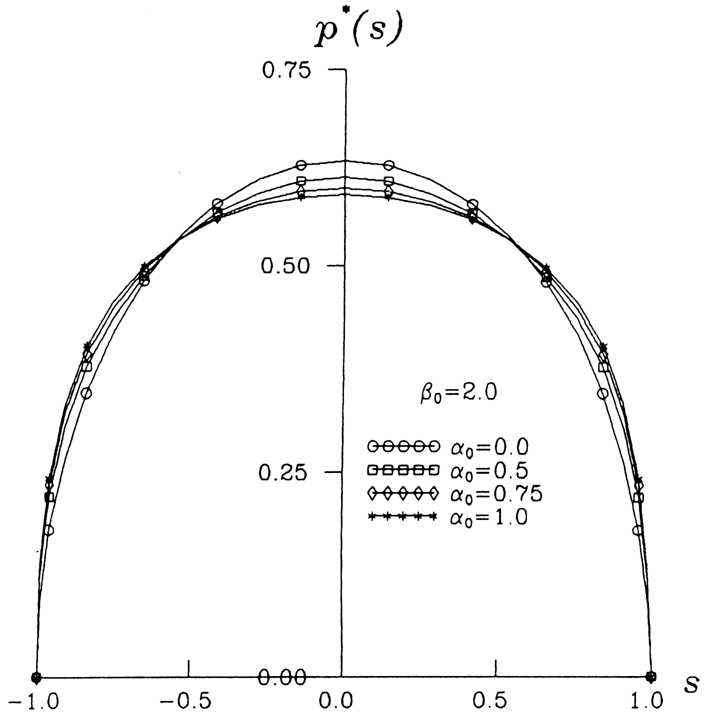

Fig. 3. Normalized contact pressure $p^*(s)$ as a function of α_0 at fixed value β_0.

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